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NOTES ON DYNAMICS—2
CHARACTERISATION OF DYNAMICAL SYSTEMS

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NOTES ON DYNAMICS—2
CHARACTERISATION OF DYNAMICAL SYSTEMS

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1. *Introduction*

Considered as a special form of the general system of differential equations of dynamics

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots\ldots = \frac{dx_k}{X_k} = dt \tag{1}$$

the canonical system

$$\frac{dq_r}{dt} = \frac{\partial H}{\partial p_r};\ \frac{dp_r}{dt} = -\frac{\partial H}{\partial q_r} \cdot (r = 1, 2, \ldots. n) \tag{2}$$

has characteristic properties like the possession of special types of integral invariants, the validity of Poisson's theorem regarding the derivation of a third integral from two given ones, the group property of integrals, constancy of Langrange bracket expressions along a trajectory, etc.

We consider in this note the converse problem of showing that the general system (1) taken in the suitable form

$$\left.\begin{aligned} \frac{dq_r}{dt} &= A_r(q_1, \ldots. q_n, p_1, \ldots. p_n) \\ \frac{dp_r}{dt} &= B_r(q_1, \ldots. q_n, p_1 \ldots. p_n) \end{aligned}\right\} \quad (r = 1, 2, \ldots. n) \tag{3}$$

is necessarily of the canonical form (2) when any of the above characteristic properties is satisfied. This has been proved in a few cases, but a systematic examination of the question brings out several interesting points which deserve to be placed on record.

2. *Integral Invariants*

The canonical system (2) possesses the set of n integral invariants whose integrands

$$\Omega^{(1)},\ \Omega^{(2)},\ \ldots.\ \Omega^{(n)}$$

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are built from (Prange, p. 668)

$$\Omega^{(1)} = \Omega(\lambda, \mu) = \Sigma(\delta^{(\lambda)} p_\rho \, \delta^{(\mu)} q_\rho - \delta^{(\lambda)} q_\rho \, \delta^{(\mu)} p_\rho) \tag{4}$$

by the method of exterior multiplication of alternating differential forms (Cartan) or, what is equivalent to the same thing, the symbolic product of such differential forms (Goursat). In the usual notation of the integral calculus the first of this set of integral invariants can be written as

$$\int\int \Sigma \, \delta q_i \, \delta p_i = \text{Const.} \tag{4'}$$

the general one of order $2r$ as

$$\underbrace{\int\int \cdots \int}_{2r} \Sigma \, \delta q_{i_1} \ldots \delta q_{i_r} \, \delta p_{i_1} \ldots \delta p_{i_r} = \text{Const.} \tag{5}$$

and the last one as

$$\underbrace{\int\int \cdots \int}_{2n} \delta q_1 \ldots \delta q_n \, \delta p_1 \ldots \delta p_n = \text{Const.} \tag{6}$$

It is well known that systems possessing the integral invariant (4′) are necessarily of the canonical form (Whittaker, p. 272). We now proceed to show that the same method of proof can be extended to prove the result for systems possessing the integral invariant (5). Taking the system in the form (3) the condition of integral invariancy of (5) leads to

$$\underbrace{\int\int \cdots \int}_{2r} \Sigma \, \frac{\partial(q_{i_1}, \ldots q_{i_r}, p_{i_1}, \ldots p_{i_r})}{\partial(\lambda_1 \ldots \lambda_r, \mu_1 \ldots \mu_r)} = \text{Const.}$$

where $\lambda_1, \ldots \lambda_r$; $\mu_1, \ldots \mu_r$ are $2r$ parameters specifying any point in the domain of integration, and do not vary with the time, but are characteristic of the trajectory on which the point in question lies. We should therefore have

$$\frac{d}{dt} \Sigma \frac{\partial(q_{i_1}, \ldots q_{i_r}, p_{i_1}, \ldots p_{i_r})}{\partial(\lambda_1, \ldots \lambda_r, \mu_1, \ldots \mu_r)} = 0$$

Using the rule of differentiation of a determinant, and the equations of the system (3) we can expand this expression, and typical terms of the expansion are

$$\frac{\partial A_{i_s}}{\partial q_{i_t}} \frac{\partial(q_{i_1}, \ldots q_{i_{s-1}}, q_{i_t}, q_{i_{s+1}}, \ldots q_{i_r}, p_{i_1}, \ldots p_{i_r})}{\partial(\lambda_1, \ldots\ldots\ldots\ldots \lambda_r, \mu_1, \ldots \mu_r)}$$

$$\frac{\partial A_{i_s}}{\partial p_{i_t}} \frac{\partial(q_{i_1}, \ldots q_{i_{s-1}}, p_{i_t}, q_{i_{s+1}}, \ldots q_{i_r}, p_{i_1}, \ldots p_{i_r})}{\partial(\lambda_1, \ldots\ldots\ldots\ldots \lambda_r, \mu_1, \ldots \mu_r)}$$

$$\frac{\partial B_{i_s}}{\partial q_{i_t}} \frac{\partial (q_{i_1}, \ldots q_{i_r}, p_{i_1}, \ldots p_{i_{s-1}}, q_{i_t}, p_{i_{s+1}}, \ldots p_{i_r})}{\partial (\lambda_1, \ldots \lambda_r, \mu_1, \ldots \ldots \ldots \mu_r)}$$

$$\frac{\partial B_{i_s}}{\partial p_{i_t}} \frac{\partial (q_{i_1}, \ldots q_{i_r}, p_{i_1}, \ldots p_{i_{s-1}}, p_{i_t}, p_{i_{s+1}}, \ldots p_{i_r})}{\partial (\lambda_1, \ldots \lambda_r, \mu_1, \ldots \ldots \ldots \mu_r)}$$

In the actual expansion, summation is implied over repeated indices, so that we can take the first and fourth terms together, and write them as

$$\left(\frac{\partial A_{i_s}}{\partial q_{i_t}} + \frac{\partial B_{i_t}}{\partial p_{i_s}}\right) \frac{q (q_{i_1}, \ldots q_{i_{s-1}}, q_{i_t}, q_{i_{s+1}}, \ldots q_{i_r}, p_{i_1}, \ldots p_{i_r})}{\partial (\lambda_1, \ldots \ldots \ldots \lambda_r, \mu_1, \ldots \mu_r)}$$

Owing to the completely arbitrary natuie of the domain of integration, and the choice of the parameters $\lambda_1, \ldots \lambda_r, \mu_1, \ldots \mu_r$ this leads to

$$\frac{\partial A_{i_s}}{\partial q_{i_t}} + \frac{\partial B_{i_s}}{\partial q_{i_t}} = 0 \qquad (i_s, i_t = 1, 2, \ldots n) \tag{7}$$

Again, in the second typical term we can interchange the dummy indices i_s and i_t and the Jacobian factor of the term becomes

$$\frac{\partial (\partial_{i_1}, \ldots \partial_{i_{s-1}}, p_{i_s}, q_{i_{s+1}}, \ldots q_{i_r}, p_{i_1}, \ldots p_{i_{s-1}}, p_{i_t}, p_{i_{s+1}}, \ldots p_{i_r})}{\partial (\lambda_1, \ldots \ldots \ldots \lambda_r, \mu_1, \ldots \ldots \ldots \mu_r)}.$$

and interchange of p_{i_s} with p_{i_t} in this determinant means an interchange of columns equal in number to

$$(s-1) + (r-s+1) + (r-s) + (s-1) = 2r-1$$

i.e., an odd number. Hence equating the coefficient of

$$\frac{\partial p_{i_t}}{\partial \lambda_t} \cdot \frac{\partial p_{i_s}}{\partial \mu_s} \cdots$$

to zero, we get

$$\frac{\partial A_{i_s}}{\partial p_{i_t}} - \frac{\partial A_{i_t}}{\partial p_{i_s}} = 0 \qquad (i_s, i_t = 1, 2, \ldots n) \tag{8}$$

Treating the third typical term in the same manner yields

$$\frac{\partial B_{i_s}}{\partial q_{i_t}} - \frac{\partial B_{i_t}}{\partial q_{i_s}} = 0 \qquad (i_s, i_t = 1, 2, \ldots n) \tag{9}$$

Equations (7), (8) and (9) show that the system (3) is necessarily of the canonical form (2).

It is interesting to observe that this method fails when applied to the last integral invariant of the set, *viz*. (6). The condition for this being an integral invariant is that

$$\frac{d}{dt} \frac{\partial (q_1, \ldots q_n, p_1, \ldots p_n)}{\partial (\lambda_1, \ldots \lambda_n, \mu_1, \ldots \mu_n)} = 0$$

without any summation, and instead of conditions (7) to (9) we obtain here the single condition

$$\Sigma\left(\frac{\partial A_i}{\partial q_i}+\frac{\partial B_i}{\partial p_i}\right)=0 \tag{10}$$

and it is not immediately obvious that this is sufficient to ensure the canonical form of (3). (10) can be immediately recognised as the condition to be satisfied in order that the system (3) should possess a last multiplier M which is equal to unity, a property which is characteristic of the canonical system. Thus the above direct method does not enable us to show that the system (3) is necessarily canonical in form if it possesses a last multiplier equal to unity, or in other words if it satisfies Liouville's theorem on the invariance of volume in phase space.

In this special case, however, we can adopt an alternative method used in a well-known theorem relating to the symbolic product of two differential forms. The theorem in question is that if Ω_p and Ω_q be two differential forms of orders p and q and if $\int \Omega_p$ and $\int \Omega_q$ be integral invariants, then $\int \Omega_p\Omega_q$ is also an integral invariant, where $\Omega_p\Omega_q$ is the symbolic product (Goursat, p. 230). This theorem is proved by changing the given system of equations under the transformation due to Poincaré (Poincaré, p. 30) into

$$\frac{dy_1}{0}=\ldots\ldots=\frac{dy_{n+1}}{0}=\frac{dz}{1}=dt \tag{11}$$

so that Ω_p and Ω_q transform themselves into the symbolic forms Ω_p', Ω_q'. Since Ω_p and Ω_q are integrands of integral invariants, the coefficients of Ω_p' and Ω_q' do not involve z, and the same is true of $\Omega_p'\cdot\Omega_q'$. Hence $\Omega_p\Omega_q$ of which $\Omega_p'\Omega_q'$ is the transform, is itself the integrand of an invariant integral. This theorem can be generalised to the symbolic product of more than two differential forms, and in particular shows that $\int(\Omega_p)^r$ is also an integral invariant if $\int \Omega_p$ be so. The converse of the theorem is not true in general, but in the above particular case the converse holds since the assumption that the coefficients of $(\Omega'_p)^r$ do not involve z necessarily leads to the conclusion that the coefficients of Ω_p' also do not. In the symbolic notation the integrand of (6), *viz.*, $\Omega^{(n)}$ can be written as $(\Omega^{(1)})^n$, and hence if $\int \Omega^{(n)}$ be an integral invariant it follows that the same is true of $\int \Omega^{(1)}$ which in turn leads to the conclusion that the system of equations must be of the canonical form.

The method of proof is obviously applicable to every $\Omega^{(r)}$ $(r=1,2,\ldots.n)$.

3. *Poisson's Theorem*

The result that the system (3) is canonical provided the existence of two integrals $f_1=c_1$, $f_2=c_2$ demands that $(f_1, f_2)=c$ is also an integral is due

Korkine (see also Whittaker, p. 337), and can be looked upon as the converse of Poisson's theorem. Since the paper by Korkine is inaccessible, we append below two proofs of this theorem.

(i) The proof in Goursat (*ibid.*, p. 232) of Poisson's theorem can be employed to deduce the converse also. The following is the proof of the direct theorem using symbolic products.

Consider the integral invariant

$$\int \Omega^{(1)} \text{ or } \int \Omega_1 = \int \sum_i dq_i\, dp_i \tag{4'}$$

Then $(\Omega_1)^n \equiv n\,!\, dq_1\, dp_1 \ldots . dq_n\, dp_n$. If f_1 and f_2 be two functions of (q_i, p_i), the symbolic product $(\Omega_1)^{n-1}\, df_1\, df_2$ is equal to $(f_1, f_2)\, dq_1\, dp_1 \ldots . dq_n\, dp_n$ (*ibid.*, p. 164) except for a constant factor. Now if f_1 and f_2 be two integrals of the canonical system, df_1 and df_2 are solutions of the variational equations of the system, and $\int \Omega_1$ being an integral invariant, so also are $\int (\Omega_1)^n$ and $\int (\Omega_1)^{n-1}$. Thus $(\Omega_1)^n$ and $(\Omega_1)^{n-1}\, df_1\, df_2$ give integrals of the variational system and so does

$$\frac{(\Omega_1)^{n-1}\, df_1\, df_2}{(\Omega_1)^n} \tag{12}$$

i.e., (f_1, f_2) is an integral of the variational system, and since it does not involve the differentials it is a solution of the original system. This is Poisson's theorem.

To adapt this to prove the converse we note that, if we are given that (f_1, f_2) is an integral whenever f_1 and f_2 are integrals, it follows that (12) is a solution of the variational equations. Since f_1, f_2 are integrals df_1 and df_2 are solutions of the variational equations, and the same is therefore true of Ω_1, *i.e.*, $\int \Omega_1$ is an integral invariant, and the system is canonical.

(ii) We can however also give a direct proof:—

If $f_1 = c_1$ and $f_2 = c_2$ be integrals of (3), $\dfrac{df_1}{dt} = 0$, $\dfrac{df_2}{dt} = 0$ give, using the summation convention

$$\frac{df_1}{dt} + \mathrm{A}_k \frac{df_1}{dq_k} + \mathrm{B}_k \frac{\partial f_1}{dp_k} = 0 \tag{13}$$

$$\frac{df_2}{dt} + \mathrm{A}_k \frac{df_2}{dq_k} + \mathrm{B}_k \frac{df_2}{dp_k} = 0 \tag{14}$$

Similarly if $(f_1, f_2) =$ Const. be an integral

$$\frac{\partial (f_1, f_2)}{dt} + \mathrm{A}_k \frac{d(f_1, f_2)}{\partial q_k} + \mathrm{B}_k \frac{\partial (f_1, f_2)}{\partial p_k} = 0 \tag{15}$$

or,

$$\left(\frac{\partial f_1}{\partial t}, f_2\right)+\left(f_1, \frac{\partial f_2}{\partial t}\right)+A_k\left(\frac{\partial f_1}{\partial q_k}, f_2\right)+A_k\left(f_1, \frac{\partial f_2}{\partial q_k}\right)$$
$$+B_k\left(\frac{\partial f_1}{\partial p_k}, f_2\right)+B_k\left(f_1, \frac{\partial f_2}{\partial p_k}\right)=0 \qquad (15')$$

We now take the Poisson bracket of (13) with f_2 on the right, that of (14) with f_1 on the left, add the results and subtract (15′) from this sum. These operations give

$$\frac{\partial f_1}{\partial q_k}(A_k, f_2)+\frac{\partial f_1}{\partial p_k}(B_k, f_2)+\frac{\partial f_2}{\partial q_k}(f_1, A_k)+\frac{\partial f_2}{\partial p_k}(f_1, B_k)=0 \qquad (16)$$

Writing the expanded form of the Poisson brackets, and collecting coefficients of $\frac{\partial f_1}{\partial q_k}\frac{\partial f_2}{\partial q_j}$, $\frac{\partial f_1}{\partial q_k}\frac{\partial f_2}{\partial p_j}$, $\frac{\partial f_1}{\partial p_k}\frac{\partial f_2}{\partial q_j}$, $\frac{\partial f_1}{\partial p_k}\frac{\partial f_2}{\partial p_j}$ after suitable interchange of dummy indices we get

$$\frac{\partial f_1}{\partial q_k}\frac{\partial f_2}{\partial q_j}\left(\frac{\partial A_k}{\partial p_j}-\frac{\partial A_j}{\partial p_k}\right)+\frac{\partial f_1}{\partial p_k}\frac{\partial f_2}{\partial q_j}\left(\frac{\partial B_k}{\partial p_j}+\frac{\partial A_j}{\partial q_k}\right)$$
$$-\frac{\partial f_1}{\partial q_k}\frac{\partial f_2}{\partial p_j}\left(\frac{\partial A_k}{\partial q_j}+\frac{\partial B_j}{\partial p_k}\right)-\frac{\partial f_1}{\partial p_k}\frac{\partial f_2}{\partial p_j}\left(\frac{\partial B_k}{\partial q_j}-\frac{\partial B_j}{\partial q_k}\right)=0 \qquad (17)$$

Since f_1, f_2 are arbitrary we get by equating to zero the coefficients of $\frac{\partial f_1}{\partial q_k}\frac{\partial f_2}{\partial q_j}$, etc.

$$\left.\begin{aligned}\partial A_k/\partial p_j-\partial A_j/\partial p_k&=0\\ \partial B_k/\partial q_j-\partial B_j/\partial q_k&=0\\ \partial A_j/\partial q_k+\partial B_k/\partial p_j&=0\end{aligned}\right\} \qquad (18)$$

which shows that A_k, B_k have the form $\partial H/\partial p_k$ and $-\partial H/\partial q_k$.

4. *Group Property of Integrals*

We now consider whether the system is canonical if the group property of an integral is satisfied, *viz.*, if $f=c$ be an integral

$$\partial q_r=\frac{\partial f}{\partial p_r}\delta\alpha;\quad \delta p_r=-\frac{\partial f}{\partial q_r}\delta\alpha \qquad (19)$$

constitute a solution of the variational equations, and show that this is true.

Proof.—The variational equations of the system (3) are given by (Prange, p. 964).

$$\left.\begin{aligned}\frac{dx_\rho}{dt}&=\frac{\partial A_\rho}{\partial p_\lambda}\pi_\lambda+\frac{\partial A_\rho}{\partial q_\lambda}x_\lambda\\ \frac{d\pi_\rho}{dt}&=\frac{\partial B_\rho}{\partial p_\lambda}\pi_\lambda+\frac{\partial B_\rho}{\partial q_\lambda}x_\lambda\end{aligned}\right\} \qquad (20)$$

If (19) be solutions of this

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial f}{\partial p_\rho}\right) &= -\frac{\partial A_\rho}{\partial p_\lambda}\frac{\partial f}{\partial q_\lambda} + \frac{\partial A_\rho}{\partial q_\lambda}\frac{\partial f}{\partial p_\lambda} \\ \frac{d}{dt}\left(-\frac{\partial f}{\partial q_\rho}\right) &= -\frac{\partial B_\rho}{\partial p_\lambda}\frac{\partial f}{\partial q_\lambda} + \frac{\partial B_\rho}{\partial q_\lambda}\frac{\partial f}{\partial p_\lambda} \end{aligned}\right\} \tag{21}$$

the summation convention being implied. Also since $f = \text{Const.}$ is an integral

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_\lambda} A_\lambda + \frac{\partial f}{\partial p_\lambda} B_\lambda = 0 \tag{22}$$

(21) can be written as

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial p_\rho}\right) + \frac{\partial}{\partial q_\lambda}\left(\frac{\partial f}{\partial p_\rho}\right) A_\lambda + \frac{\partial}{\partial p_\lambda}\left(\frac{\partial f}{\partial p_\rho}\right) B_\lambda \\ &\qquad = -\frac{\partial A_\rho}{\partial p_\lambda}\frac{\partial f}{\partial q_\lambda} + \frac{\partial A_\rho}{\partial q_\lambda}\frac{\partial f}{\partial p_\lambda} \\ &\frac{\partial}{\partial t}\left(-\frac{\partial f}{\partial q_\rho}\right) + \frac{\partial}{\partial q_\lambda}\left(-\frac{\partial f}{\partial q_\rho}\right) A_\lambda + \frac{\partial}{\partial p_\lambda}\left(-\frac{\partial f}{\partial q_\rho}\right) B_\lambda \\ &\qquad = -\frac{\partial B_\rho}{\partial p_\lambda}\frac{\partial f}{\partial q_\lambda} + \frac{\partial B_\rho}{\partial q_\lambda}\frac{q f}{\partial p_\lambda} \end{aligned}\right\} \tag{23}$$

Again differentiating (22) partially w.r.t. p_ρ and $-q_\rho$, we have

$$\left.\begin{aligned} &\frac{\partial}{\partial p_\rho}\left(\frac{\partial f}{\partial t}\right) + \frac{\partial}{\partial p_\rho}\left(\frac{\partial f}{\partial q_\lambda}\right) A_\lambda + \frac{\partial f}{\partial q_\lambda}\frac{\partial A_\lambda}{\partial p_\rho} + \frac{\partial}{\partial p_\rho}\left(\frac{\partial f}{\partial p_\lambda}\right) B_\lambda \\ &\qquad + \frac{\partial f}{\partial p_\lambda}\frac{\partial B_\lambda}{\partial p_\rho} = 0 \\ &-\frac{\partial}{\partial q_\rho}\left(\frac{\partial f}{\partial t}\right) - \frac{\partial}{\partial q_\rho}\left(\frac{\partial f}{\partial q_\lambda}\right) A_\lambda - \frac{\partial f}{\partial q_\lambda}\frac{\partial A_\lambda}{\partial q_\rho} = \frac{\partial}{\partial q_\rho}\left(\frac{\partial f}{\partial p_\lambda}\right) B_\lambda \\ &\qquad - \frac{\partial f}{\partial p_\lambda}\frac{\partial B_\lambda}{\partial q_\rho} = 0 \end{aligned}\right\} \tag{24}$$

(23) and (24) give by subtraction and rearrangement

$$\left.\begin{aligned} \frac{\partial f}{\partial q_\lambda}\left(\frac{\partial A_\lambda}{\partial p_\rho} - \frac{\partial A_\rho}{\partial p_\lambda}\right) + \frac{\partial f}{\partial p_\lambda}\left(\frac{\partial A_\rho}{\partial q_\lambda} + \frac{\partial B_\lambda}{\partial p_\rho}\right) = 0 \\ \frac{\partial f}{\partial q_\lambda}\left(\frac{\partial B_\rho}{\partial p_\lambda} + \frac{\partial A_\lambda}{\partial q_\rho}\right) + \frac{\partial f}{\partial p_\lambda}\left(\frac{\partial B_\lambda}{\partial q_\rho} - \frac{\partial B_\rho}{\partial q_\lambda}\right) = 0 \end{aligned}\right\} \tag{25}$$

Since f is entirely arbitrary we equate the coefficients of $\partial f/\partial q_\lambda$ and $\partial f/\partial p_\lambda$ to zero in both the equations of (25) and derive the relations (18) which prove that (3) is a canonical system.

5. *Canonical Transformations*

We give here several theorems related to canonical transformations and giving the conditions under which the system (3) is canonical.

THEOREM 1.—If the motion along a trajectory of the system is denoted by an infinitesimal canonical transformation the system is canonical.

This is evident from the equations of an infinitesimal canonical transformation, *viz.*,

$$\Delta q_r = \frac{\partial K}{\partial p_r}\,\Delta t;\quad \Delta p_r = -\,\frac{\partial K}{\partial q_r}\Delta t$$

THEOREM 2.—The constancy of the Lagrange bracket expressions along a trajectory implies that the system is canonical.

If $f_i = a_i$, $f_j = a_j$ be two of the $2n$ integrals of the system, the constancy of the Lagrange bracket expression

$$\frac{d}{dt}\,[a_i,\ a_j] = 0$$

means

$$\frac{d}{dt}\left(\frac{\partial q_r}{\partial a_i}\frac{\partial p_r}{\partial a_j} - \frac{\partial q_r}{\partial a_j}\frac{\partial p_r}{\partial a_i}\right) = 0$$

and hence

$$\frac{d}{dt}\left(\frac{\partial q_r}{\partial a_i}\frac{\partial p_r}{\partial a_j} - \frac{\partial q_r}{\partial a_j}\frac{\partial p_r}{\partial a_i}\right)\,\Delta a_i\,\delta a_j = 0$$

i.e.,

$$\frac{d}{dt}\Big(\Delta\, q_r\,\delta p_r - \delta q_r\,\Delta\, p_r = 0\Big)$$

or the transformation of the co-ordinates from t to $t + dt$ is an infinitesimal canonical transformation, and from Theorem 1 it follows that the system is canonical.

THEOREM 3.—If the original system be transformed under a canonical transformation to a canonical system, the original system is canonical. Let the canonical transformation be

$$p_\rho = \phi_\rho\,(P_1, \ldots P_n, Q_1 \ldots Q_n);\ q_\rho = \psi_\rho\,(P_1, \ldots P_n, Q_1, \ldots Q_n) \quad (26)$$

with

$$(\phi_\sigma,\ \phi_\tau) = 0,\ (\psi_\sigma,\ \psi_\tau) = 0,\ (\phi_\sigma,\ \psi_\tau) = \delta_{\sigma\tau} \quad (27)$$

From this follow the relations between the Lagrange brackets

$$[\phi_\sigma,\ \phi_\tau] = 0,\ [\psi_\sigma,\ \psi_\tau] = 0,\ [\phi_\sigma,\ \psi_\tau] = \delta_{\sigma\tau}$$

and this means that the transformation from $(P_\rho,\ Q_\rho)$ to $(p_\rho,\ q_\rho)$ is canonical and since the system in $(P_\rho,\ Q_\rho)$ is canonical the same is true of the original one because the canonical form is preserved under canonical transformations.

This theorem is merely a consequence of the group property that the inverse of the element of a group also belongs to the group.

THEOREM 4.—If the original system becomes canonical under the transformation (26) the transformation is not necessarily canonical.

If (3) reduces under (26) to

$$\frac{dQ_r}{dt} = A_r' ; \quad \frac{dP_r}{dt} = B_r'$$

the conditions that this system be canonical involve, in general, the A_r and B_r of the system (3), and thus in order that the transformation should satisfy (27) without involving A_r, B_r these latter must be of special forms, *i.e.*, the given system must be of a special type. It can be shown that for the following system the transformation must be canonical.

(i) It must be the Pfaffian system of a Pfaffian differential form whose bilinear covariant is

$$\sum_{1}^{2n+1} \sum_{1}^{2n+1} C_{\rho\sigma} (du_\rho \, \delta u_\sigma - \delta u_\rho \, du_\sigma) \quad \left(C_{\rho\sigma} = \frac{\partial U_\rho}{\partial u_\sigma} - \frac{\partial U_\sigma}{q u_\rho} \right).$$

(ii) The transformation must reduce the bilinear covariant to the normal form

$$\Sigma \, \Sigma \, (du_\rho \, \delta u_\sigma - \delta u_\rho \, du_\sigma).$$

This reduction is analogous to the reduction of a general quadratic form $\Sigma \Sigma g_{\rho\sigma} \, du_\rho \, du_\sigma$ to the normal form $\Sigma \, du_\sigma^2$, and for this it is necessary that the curvature tensor should vanish. Similar results must also hold for $C_{\rho\sigma}$ (*cf.* Prange, p. 756, footnote).

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